Recall the following summation identity:

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 1
Let $f : [0,1] \to \mathbb{R}, f(x) = x^2$, and $P = \{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$. Compute $U(f,P)$ and $L(f,P)$. Conclude that f is integrable.

Problem 2
Let $f : [a,b] \to \mathbb{R}$ be an integrable function and suppose that $[c,d] \subseteq [a,b]$. Show that the restriction of f to $[c,d]$ is also integrable.

Problem 3
Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that f is integrable. Generalize this result to $[a,b]$ instead of $[0,1]$. *Hint: Recall that continuous over a closed interval implies uniformly continuous.*

Problem 4

1. Let $f : [a,b] \to \mathbb{R}$ be continuous, except at one point $c \in [a,b]$. Also suppose f is bounded. Show that f is still integrable.
2. Let $f : [a,b] \to \mathbb{R}$ be continuous, except at finitely many points $c_1, \dots, c_n \in [a,b]$. Also suppose f is bounded. Using induction, show that f is still integrable.

Problem 5
Recall Thomae's function (restricted to $[0,1]$)

$$f : [0,1] \to \mathbb{R}, f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms}^{a} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Show that f is integrable, and $\int f = 0$.

[a] For $x = 0$, we define $f(x) = 1$ by convention, as f is 1 at any other integer.